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Intertemporally dependent preferences and optimal dynamic behavior

Tapan Mitra* and Kazuo Nishimura†

We study a model of optimal dynamic behavior in which the intertemoral preferences preserve the time additively separable framework of Ramsey models, while exhibiting Edgeworth–Pareto complementarity between consumption in adjacent periods. We identify economic environments in which global optimal dynamics under intertemporal complementarity exhibits persistent fluctuations even though the misspecified Ramsey-type theory, under the intertemporal independence assumption, predicts monotone convergence.

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1 Introduction

The theory of optimal intertemporal allocation has been developed primarily for the case in which the objective function of the planner or representative agent can be written as:

$$U(c_1,\ c_2 \ldots) \equiv \sum_{t=1}^{\infty} \delta^{t-1} w(c_t), \tag{1}$$

where c_t represents consumption at date t, w the period felicity function, and $\delta \in (0, 1)$ a discount factor, representing the time preference of the agent.[1] An objective function like (1) leads to the study of the following problem of Ramsey-optimal growth (under positive

*Department of Economics, Cornell University, Ithaca, NY, USA. Email: tm19@cornell.edu
†Institute of Economic Research, Kyoto University, Kyoto, Japan.

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[1] When future utilities are not discounted, programs are compared using some version of the Ramsey–Atsumi–Weizsacker overtaking criterion, but the intertemporal independence aspect of preferences over time is still kept intact. For a recent contribution, which uses this preference structure and treats the undiscounted case, see Khan and Mitra (2005).

discounting) in the standard aggregative model, with a production function, f:

$$\left.\begin{array}{l} \textit{Maximize } \sum_{t=1}^{\infty} \delta^{t-1} w(c_t) \\ \textit{subject to } c_{t+1} = f(k_t) - k_{t+1} \;\; for \;\; t \geq 0 \\ (c_t, k_t) \geq 0 \;\; for \;\; t \geq 1, k_0 = k > 0 \end{array}\right\}. \tag{2}$$

The restrictive form of the objective function (1) has been criticized on the ground that it ignores the intertemporal dependence in the preference structure. One of the first full-fledged alternative formulations to (1) is contained in the paper by Henry Wan (1970). In his formulation, intertemporal dependence is explicitly introduced in the objective function, and the strength of the complementarity between consumption at two dates is postulated to geometrically decline as the dates get more distant from each other.[2]

Two interesting features emerge in the study of Wan (1970). First, the misspecified model (2) can give quite different predictions of optimal dynamic behavior compared to the correctly specified one, incorporating intertemporal complementarity. Second, a turnpike behavior of optimal paths is observed in several concrete examples in his framework.

Samuelson (1971) proposes a simpler model in which one can investigate thoroughly the circumstances under which the turnpike property continues to hold even under intertemporal complementarity. In his formulation, the felicity derived by the agent in period $(t+1)$ depends on consumption in that period (c_{t+1}), and also on past consumption (c_t), so that the objective function becomes:

$$U(c_1, \; c_2, \ldots) = \sum_{t=1}^{\infty} \delta^{t-1} w(c_t, \, c_{t+1}). \tag{3}$$

The corresponding dynamic optimization exercise (under discounting) can be written as:

$$\left.\begin{array}{l} \textit{Maximize } \sum_{t=1}^{\infty} \delta^{t-1} w(c_t, c_{t+1}) \\ \textit{subject to } c_{t+1} = f(k_t) - k_{t+1} \;\; for \;\; t \geq 1 \\ (c_{t+1}, k_{t+1}) \geq 0 \;\; for \;\; t \geq 1, (k_1, c_1) = (k, c) > 0 \end{array}\right\}. \tag{4}$$

Problem (4) is now viewed as the standard optimization problem under "habit formation". In this interpretation, the felicity derived by the agent in period $(t+1)$ depends on consumption in that period (c_{t+1}), but the felicity function itself is (endogenously) determined by past consumption (c_t).

Samuelson (1971) provides an analysis of the circumstances under which the turnpike property would hold locally for the dynamic optimization problem. Boyer (1978) provides a more general local analysis, indicating the circumstances under which the stationary optimal stock would be unstable. It seems to us to that it is desirable to have an analysis of problem (4) that would be able to predict dynamic optimal behavior *globally*. Before we

[2] There has been considerable research on alternative formulations to (1) since the paper by Wan (1970). We deliberately refrain from a full literature survey here, referring the reader instead to our earlier paper (Mitra and Nishimura 2005) and the references cited there.

review our progress in this direction, it is useful to reformulate problems (2) and (4) to their reduced forms.

Problem (2) can be written in its "reduced form" as:

$$\left.\begin{array}{c} \textit{Maximize} \sum_{t=0}^{\infty} \delta^t u(x_t,\ x_{t+1}) \\ \textit{subject to}\,(x_t,\ x_{t+1}) \in \Omega \quad \textit{for}\ t \in \{0, 1, 2, \ldots\} \\ x_0 = x \end{array}\right\}, \tag{5}$$

where $\delta \in (0, 1)$ is the discount factor, X is a compact set (representing the state space), $\Omega \subset X \times X$ is a transition possibility set, $u : \Omega \to \mathbb{R}$ is a utility function, and $x \in X$ is the initial state of the system.

Similarly, problem (4) can be written in the following "reduced form":

$$\left.\begin{array}{c} \textit{Maximize} \ \ \sum_{t=0}^{\infty} \delta^t u(x_t,\ x_{t+1},\ x_{t+2}) \\ \textit{Subject to}\ (x_t,\ x_{t+1},\ x_{t+2}) \in \Lambda \quad \textit{for}\ t \in \{0, 1, 2, \ldots\} \\ (x_0,\ x_1) = (x, y) \end{array}\right\}, \tag{6}$$

where $\delta \in (0, 1)$ is the discount factor, X is a compact set, $\Omega \subset X \times X$ is a transition possibility set, $\Lambda = \{(x, y, z) : (x, y) \in \Omega \text{ and } (y, z) \in \Omega\}$, $u : \Lambda \to \mathbb{R}$ is a utility function, and $(x, y) \in \Omega$ is the initial state of the system.

Notice that even under intertemporal dependence in tastes, we have a recursive structure in the dynamic optimization problem (6) very much like in (5). The difference is that the state space is X in problem (5), whereas it is a subset of X^2 in problem (6). Therefore, for problem (5), (optimal) value and policy functions are defined on X, and for problem (6), these functions are defined on $\Omega \subset X^2$. In terms of examining the dynamic behavior of optimal programs, we are, therefore, dealing with a one-dimensional dynamical system for problem (5) and a two-dimensional dynamical system for problem (6).

In Mitra and Nishimura (2005), we examine problem (4) in its reduced form (6). It is observed there that if the intertemporal complementarity, expressed in the magnitude of w_{12}, is negative and small (in absolute value) relative to the direct second-order effects (w_{11} and w_{22}), then the reduced-form utility function is supermodular on Λ. It can then be shown that the value function is supermodular on Ω, and the policy function is increasing in both arguments on Ω. Furthermore, the ϕ-policy function, defined on X, by

$$\phi(x) = h(x, x) \qquad \textit{for}\ \ x \in X$$

is seen to satisfy a "single-crossing condition". These properties allow us to establish a global turnpike property, thereby providing the global counterpart to Samuelson's analysis, and contributing to the second theme to emerge from the paper by Wan (1970).

In the current paper, we focus on the first theme in Wan (1970). Specifically, our objective is to identify economic environments in which global optimal dynamics under intertemporal complementarity exhibit persistent fluctuations even though the misspecified Ramsey-type theory, under the intertemporal independence assumption, predicts monotone convergence.

In terms of the specification of the model, the only difference from Mitra and Nishimura (2005) is that intertemporal complementarity, as measured by w_{12}, is now assumed to be positive. This allows us to establish fairly clear-cut global dynamic behavior, which is shown to be one of two types: either there is global convergence to the stationary optimal stock, or there are persistent fluctuations around the stationary optimal stock.

We use local analysis near the stationary optimal stock to identify the circumstances under which it is locally a repellor. This is seen to occur under a curvature condition, and a condition that requires that the intertemporal dependence be strong: w_{12} is not small relative to direct second-order effects ($|w_{11}|$ and $|w_{22}|$). Under these circumstances the first type of global dynamic behavior can be ruled out, and the second type of behavior holds globally. These conditions, under which volatility is introduced into the standard aggregative model, do not involve non-convexities of any sort, and they do not involve low discount factors.

2 The framework

2.1 A model of optimal growth with intertemporally dependent preferences

The model (which follows Samuelson (1971) and Boyer (1978) closely) can be described in terms of a production function, f, a welfare function, w, and a discount factor, δ.

The production function, f, is a function from $\mathbb{R}_+$ to itself which satisfies:

(f) $f(0) = 0; f$ is increasing, strictly concave and continuous on $\mathbb{R}_+$ with $\sup_{x>0}[f(x)/x] > 1$ and $\inf_{x>0}[f(x)/x] < 1$. Furthermore, f is twice continuously differentiable on $\mathbb{R}_{++}$ with $f'(x) > 0$ and $f''(x) < 0$ for all $x > 0$.

The welfare function, w, is a function from $\mathbb{R}^2_+$ to $\mathbb{R}$, which satisfies:

(w) w is continuous and strictly concave on $\mathbb{R}^2_+$; it is increasing in both arguments.[3] Furthermore, w is twice continuously differentiable on $\mathbb{R}^2_{++}$ with $w_1(c, d) > 0$, $w_2(c, d) > 0$, the Hessian of w negative definite, and $w_{12}(c, d) > 0$ for every $(c, d) \in \mathbb{R}^2_{++}$.

In what follows, we normalize $w(0, 0)$ to 0.

The discount factor, δ, is assumed to satisfy:

(d) $0 < \delta < 1$, and $\sup_{x>0}[\delta f(x)/x] > 1$.

The second part of (d) is a familiar δ-productivity assumption.

Under assumption (f), it is well known that there is a unique positive solution to the equation: $f(x) = x$. We denote this solution by B, and note that for $0 < x < B$, we have $B > f(x) > x$, whereas for $x > B$, we have $B < f(x) < x$.

A program, in this framework, is described by a sequence (k_t, c_t), where k_t denotes the capital stock and c_t the consumption in period t. The initial condition is specified by $(k, c) \geq 0$.

[3] Boyer (1978) assumes that w is increasing in both arguments. Samuelson (1971) does not; he assumes instead that $w(c, c)$ is increasing in c. It is this latter assumption that is crucial in proving the uniqueness of a stationary optimal stock in this model and, therefore, of our "single-crossing property".

Formally, a program (k_t, c_t) from (k, c) is a sequence satisfying

$$\left.\begin{aligned} (k_1,\ c_1) = (k, c),\ k_{t+1} = f(k_t) - c_{t+1} \qquad & \textit{for } t \geq 1 \\ 0 \leq c_{t+1} \leq f(k_t) \qquad & \textit{for } t \geq 1 \end{aligned}\right\}. \tag{7}$$

Note that the choice of consumption decisions, c_t, starts from $t \geq 2$.

If $(k_t,\ c_t)$ is a program from $(k,\ c) \leq (B,\ B)$, then we have $(k_t,\ c_t) \leq (B,\ B)$ for all $t \geq 1$. In what follows, we will always restrict the initial condition for programs to $(k,\ c) \leq (B,\ B)$, without further mention. The set $[0,\ B]$ becomes the natural state space for our model, and we denote this set by X.

An optimal program from (k, c) is a program $(\bar{k}_t,\ \bar{c}_t)$ satisfying

$$\sum_{t=1}^{\infty} \delta^{t-1} w(c_t,\ c_{t+1}) \leq \sum_{t=1}^{\infty} \delta^{t-1} w(\bar{c}_t,\ \bar{c}_{t+1}) \tag{8}$$

for every program (k_t, c_t) from (k, c).

2.2 Conversion to reduced form

It is covenient for our analysis of the model described above to convert it to its reduced form, where one keeps track only of the capital stock sequence.

To reformulate the optimality exercise in (8) subject to (7), we can proceed as follows. Define a transition possibility set, Ω, as:

$$\Omega = \{(x,\ y) : x \in X,\ 0 \leq y \leq f(x)\}.$$

It is easy to check that the transition possibility set, Ω, is a subset of X^2, satisfying

Assumption 1 $(0, 0)$ and (B, B) are in Ω; if $(0,\ y) \in \Omega$ then $y = 0$.

Assumption 2 Ω is closed and convex.

Assumption 3 If $(x,\ y) \in \Omega$ and $x \leq x' \leq B, 0 \leq y' \leq y$, then $(x',\ y') \in \Omega$.

Assumption 4 There is $(\tilde{x},\ \tilde{y}) \in \Omega$ with $\tilde{y} > \tilde{x}$.

Notice that for all $x \in [0,\ B]$, we have $(x,\ x) \in \Omega$. Associated with Ω is the correspondence $\Psi : X \to X$, given by $\Psi(x) = \{y : (x,\ y) \in \Omega\}$. Define the set:

$$\Lambda = \{(x,\ y, z) : (x,\ y)\ \in\ \Omega\ \textit{and}\ (y,\ z)\ \in\ \Omega\}.$$

The reduced form utility function can be defined, for $(x,\ y,\ z)$ in Λ as:

$$u(x,\ y,\ z) = w(\,f(x) - y,\ \ f(y) - z).$$

One can check that the utility function, u, is a map from Λ to $\mathbb{R}$, which satisfies:

Assumption 5 *u* is continuous and strictly concave on Λ.

Assumption 6 *u* is increasing in the first argument, and decreasing in the third argument.

On the interior of Λ, we can calculate the first-order partial derivatives of u as follows:

$$\begin{aligned}
u_1(x, y, z) &= w_1(f(x) - y,\ f(y) - z) f'(x) \\
u_2(x, y, z) &= w_2(f(x) - y,\ f(y) - z) f'(y) - w_1(f(x) - y,\ f(y) - z) \\
u_3(x, y, z) &= -w_2(f(x) - y,\ f(y) - z).
\end{aligned}$$

Because $w_1 > 0$ and $w_2 > 0$, it follows that $u_1 > 0$ and $u_3 < 0$, as required in Assumption 6.

The second-order cross partial derivatives of u can be calculated as follows:

$$\begin{aligned}
u_{12}(x, y, z) &= [w_{12}(f(x) - y,\ f(y) - z) f'(y) - w_{11}(f(x) - y,\ f(y) - z)] f'(x) \\
u_{13}(x, y, z) &= -f'(x) w_{12}(f(x) - y,\ f(y) - z) \\
u_{23}(x, y, z) &= w_{12}(f(x) - y,\ f(y) - z) - w_{22}(f(x) - y,\ f(y) - z) f'(y).
\end{aligned}$$

Because $w_{12}(c, d) > 0$ for all $(c, d) \in \mathbb{R}^2_{++}$, we have $u_{13} < 0$, $u_{12} > 0$ and $u_{23} > 0$ in the interior of Λ.

The reduced form model is then described by the triple (Ω, u, δ). We can describe programs in this model (conveying the same information as in (7) above) as follows.

The initial condition (which should be considered to be historically given) is specified by a pair (x, y) in Ω. A program (x_t) from (x, y) is a sequence satisfying

$$x_0 = x,\ x_1 = y,\ (x_t,\ x_{t+1}) \in \Omega \ \text{ for } t \geq 1. \tag{9}$$

Therefore, in specifying a program, the period 0 and period 1 states are historically given. Choice of future states starts from $t = 2$. Notice that for a program (x_t) from $(x, y) \in \Omega$, we have $(x_t, x_{t+1}, x_{t+2}) \in \Lambda$ for $t \geq 0$.

The initial condition (k, c) in (7) translates to the initial condition in (9) as $(x, y) = (f^{-1}(k_1 + c_1), k_1)$. That is, x is the capital stock (in period 0) that produced the output $(k_1 + c_1)$ in period 1, which was split up between consumption (c_1) and capital stock (k_1) in period 1; y is the capital stock in period 1.

An *optimal program* $(\bar{x}_t)$ from $(x, y) \in \Omega$ is a program from (x, y) satisfying

$$\sum_{t=0}^{\infty} \delta^t u(x_t,\ x_{t+1},\ x_{t+2}) \leq \sum_{t=0}^{\infty} \delta^t u(\bar{x}_t,\ \bar{x}_{t+1},\ \bar{x}_{t+2}) \tag{10}$$

for every program (x_t) from (x, y).

Under our assumptions, a standard argument suffices to ensure the existence of an optimal program from every initial condition $(x, y) \in \Omega$. Using Assumptions (2) and (5), it can also be shown that this optimal program is unique.

We can define a *value function*, $V : \Omega \to \mathbb{R}$ by

$$V(x, y) = \sum_{t=0}^{\infty} \delta^t u(\bar{x}_t,\ \bar{x}_{t+1},\ \bar{x}_{t+2}), \tag{11}$$

where $(\bar{x}_t)$ is the optimal program from (x, y). Then, V is concave and continuous on Ω.

It can be shown that for each $(x, y) \in \Omega$, the Bellman equation

$$V(x, y) = \max_{(y,z)\in\Omega} [u(x, y, z) + \delta V(y, z)] \tag{12}$$

holds. Also, V is the unique continuous function on Ω, which solves the functional equation (12).

For each $(x, y) \in \Omega$, we denote by $h(x, y)$ the value of z that maximizes $[u(x, y, z) + \delta V(y, z)]$ among all z satisfying $(y, z) \in \Omega$. Then, a program (x_t) from $(x, y) \in \Omega$ is an optimal program from (x, y) if and only if:

$$V(x_t,\ x_{t+1}) = u(x_t,\ x_{t+1},\ x_{t+2}) + \delta V(x_{t+1},\ x_{t+2}) \qquad \textit{for } t \geq 0. \tag{13}$$

This, in turn, holds if and only if

$$x_{t+2} = h(x_t,\ x_{t+1}) \qquad \textit{for } t \geq 0. \tag{14}$$

We will call h the (optimal) policy function. It can be shown by using standard arguments that h is continuous on Ω.

3 Basic properties of the model

3.1 Properties of a stationary optimal stock

In this subsection, we present some basic properties of a stationary optimal stock in our framework, which will be useful in studying global and local dynamics of optimal programs in the next two sections.

Given assumptions (f) and (d), there is a unique positive solution to the equation:

$$\delta f'(x) = 1. \tag{15}$$

We denote this solution by $\bar{x}$ and note that $0 < \bar{x} < B$. It follows then by (f) that:

$$f(\bar{x}) = [f(\bar{x})/\bar{x}]\bar{x} > f'(\bar{x})\bar{x} = \bar{x}/\delta \tag{16}$$

and, consequently:

$$f(\delta\bar{x}) \geq \delta f(\bar{x}) > \bar{x}. \tag{17}$$

Using (16) and (17), we have $(\delta\bar{x}, \bar{x}, \bar{x}/\delta) \in \Lambda$, and:

$$u(\delta\bar{x}, \bar{x}, \bar{x}/\delta) = w(f(\delta\bar{x}) - \bar{x},\ f(\bar{x}) - (\bar{x}/\delta)) > w(0, 0) = 0 = u(0, 0, 0).$$

This property of the reduced-form model is worth noting explicitly as:

Assumption 7 For $\bar{x}$ satisfying (15), we have $(\delta\bar{x}, \bar{x}, \bar{x}/\delta) \in \Lambda$, and $u(\delta\bar{x}, \bar{x}, \bar{x}/\delta) > u(0, 0, 0)$.

Assumption 7 is a δ-normality assumption jointly on (Λ, u, δ). It is analogous to the δ-normality assumption in the usual reduced-form model, where it is used to establish the existence of a nontrivial stationary optimal stock.

As is to be expected (from Samuelson 1971) $\bar{x}$ is the unique nontrivial stationary state of the dynamical system (14), a result that is worth stating explicitly.

Lemma 1 *If $k \in (0, B)$ satisfies $h(k, k) = k$, then $k = \bar{x}$, where $\bar{x}$ is the unique solution of* (15).

PROOF: Given that $h(k, k) = k$, we know that the sequence $(k, k, k, k, \ldots)$ is optimal starting from (k, k). Because $k \in (0, B)$, we know from (f) that $f(k) - k > 0$, so that (k, k, k) belongs to the interior of Λ. Therefore, it must satisfy the following version of the Ramsey–Euler equation for our framework:

$$u_3(k, k, k) + \delta u_2(k, k, k) + \delta^2 u_1(k, k, k) = 0.$$

Then, denoting $f(k) - k$ by c, we have:

$$w_2(c, c)(-1) + \delta[w_1(c, c)(-1) + w_2(c, c) f'(k)] + \delta^2 w_1(c, c) f'(k) = 0$$

and this yields, using (w), the result that $\delta f'(k) = 1$. Therefore, k must be equal to $\bar{x}$. □

Denote $[0, \bar{x}]$ by Y, and consider, for each $k \in X$, the following constrained optimization problem:

$$\left.\begin{array}{ll} \textit{Maximize} & u((1-\delta^2)k + \delta^2 x, (1-\delta)k + \delta x, x) \\ \textit{subject to} & x \in Y \end{array}\right\} (P).$$

Using Assumptions 3 and 7, we have, for each $k \in X$, $((1-\delta^2)k + \delta^2 x, (1-\delta)k + \delta x, x) \in \Lambda$. To see this, note that $((1+\delta)k, k, 0) \in \Lambda$ for all $k \in X$ and $(\delta x, x, x/\delta) \in \Lambda$ for all $x \in Y$, so that:

$$(1-\delta)((1+\delta)k, k, 0) + \delta(\delta x, x, x/\delta) = ((1-\delta^2)k + \delta^2 x, (1-\delta)k + \delta x, x) \in \Lambda.$$

Therefore, problem (P) is well-defined. For each $k \in X$, there is a unique solution to (P), which we denote by $g(k)$. The function, g, can be shown to be continuous on X by an application of the maximum theorem. Now, Lemma 1 leads to the following result.

Lemma 2 *For all $k \in [0, \bar{x})$, we have $g(k) > k$.*

PROOF: For $k = 0$, we have $u((1-\delta^2)k + \delta^2\bar{x}, (1-\delta)k + \delta\bar{x}, \bar{x}) = u(\delta^2\bar{x}, \delta\bar{x}, \bar{x})$. By Assumption 7, we obtain $u(\delta^2\bar{x}, \delta\bar{x}, \bar{x}) \geq \delta u(\delta\bar{x}, \bar{x}, \bar{x}/\delta) + (1-\delta)u(0, 0, 0) = \delta u(\delta\bar{x}, \bar{x}, \bar{x}/\delta) > u(0, 0, 0)$. Therefore, $x = 0$ cannot solve problem (P) when $k = 0$. This means $g(0) > 0$.

If the claim of the Lemma does not hold, then by continuity of g, there must be some $k \in (0, \bar{x})$ for which $g(k) = k$ must hold. That is, for this k, k itself must solve problem (P).

But, then we must have:

$$u(k, k, k) \geq u((1-\delta^2)k + \delta^2(k+\varepsilon), (1-\delta)k + \delta(k+\varepsilon), (k+\varepsilon)),$$

with $\varepsilon > 0$ small enough so that $(k+\varepsilon) \in Y$. This can be rewritten as:

$$w(f(k) - k,\ f(k) - k) \geq w(f(k+\delta^2\varepsilon) - (k+\delta\varepsilon),\ f(k+\delta\varepsilon) - (k+\varepsilon)).$$

Therefore, denoting $f(k) - k$ by c, we have the first order condition:

$$0 = w_1(c, c)\{f'(k)\delta^2 - \delta\} + w_2(c, c)\{f'(k)\delta - 1\}.$$

Then, using (w), we must have $\delta f'(k) = 1$, which means that $k = \bar{x}$, a contradiction. □

Using Lemmas 1 and 2, we can establish the following result.

Lemma 3 *For each* $k \in [0, \bar{x})$, *we have*:

$$u((1-\delta^2)k + \delta^2 x, (1-\delta)k + \delta x, x) \leq u(k, k, k) \textit{ for all } x \in [0, k].$$

PROOF: Suppose, contrary to the statement of the Lemma, that there is some $x \in [0, k)$ such that:

$$u((1-\delta^2)k + \delta^2 x, (1-\delta)k + \delta x, x) > u(k, k, k).$$

Note that because $k \in Y$, and $g(k) \neq k$ by Lemma 2, the uniqueness of solution to (P) implies:

$$u((1-\delta^2)k + \delta^2 g(k), (1-\delta)k + \delta g(k), g(k)) > u(k, k, k).$$

Because $g(k) > k$ by Lemma 2, we have $g(k) > k > x$, and so there is $\lambda \in (0, 1)$ such that $k = \lambda x + (1-\lambda)g(k)$. Then, we obtain:

$$\begin{aligned} u(k, k, k) &= u((1-\delta^2)k + \delta^2 k, (1-\delta)k + \delta k, k) \\ &= u(\lambda((1-\delta^2)k + \delta^2 x, (1-\delta)k + \delta x, x) \\ &\quad + (1-\lambda)((1-\delta^2)k + \delta^2 g(k), (1-\delta)k + \delta g(k), g(k))) \\ &\geq \lambda u((1-\delta^2)k + \delta^2 x, (1-\delta)k + \delta x, x) \\ &\quad + (1-\lambda)u((1-\delta^2)k + \delta^2 g(k), (1-\delta)k + \delta g(k), g(k)) \\ &> \lambda u(k, k, k) + (1-\lambda)u(k, k, k) = u(k, k, k), \end{aligned}$$

a contradiction.

3.2 Properties of the value and policy functions

In this subsection, we summarize some basic properties of the value and policy functions. The property of the value function relates to its behavior at (k, k), where $k > 0$, as k converges to 0. It is a consequence of the δ-normality Assumption 7. The proof is omitted because the method used to establish Proposition 2 in Mitra and Nishimura (2005) can be used directly.

Proposition 1 *The value function, V, satifies the property*:

$$[V(k, k)/k] \to \infty \ \ as\, k \to 0.$$

We now proceed to study two important properties of the policy function. The first is a monotonicity property that says that policy function is decreasing in the first argument. This is crucially dependent on the property of u that $u_{13} < 0$; this, in turn, relies on the assumption on w that $w_{12} > 0$.

Proposition 2 *If* $(x, y) \in \Omega$ *and* $(x', y) \in \Omega$ *and* $x' > x$, *then* $h(x', y) \leq h(x, y)$.

Proof: Let (x, y) and (x', y) belong to the interior of Ω with $x' > x$. Define $z = h(x, y)$ and $z' = h(x', y)$. We claim that $z' \leq z$. Suppose, on the contrary, that $z' > z$. We know that

$$\begin{aligned} V(x, y) &= u(x, y, z) + \delta V(y, z) \\ V(x', y) &= u(x', y, z') + \delta V(y, z'). \end{aligned}$$

Because $(x, y) \in \Omega$ and $(y, z') \in \Omega$,

$$V(x, y) \geq u(x, y, z') + \delta V(y, z').$$

Because $(x', y) \in \Omega$ and $(y, z) \in \Omega$,

$$V(x', y) \geq u(x', y, z) + \delta V(y, z).$$

Therefore, we get

$$\begin{aligned} &[u(x, y, z) + u(x', y, z')] + \delta[V(y, z) + V(y, z')] \\ &\quad \geq [u(x, y, z') + u(x', y, z)] + \delta[V(y, z') + V(y, z)]. \end{aligned} \tag{18}$$

This clearly yields:

$$u(x, y, z) + u(x', y, z') \geq u(x, y, z') + u(x', y, z).$$

This inequality can be rewritten as:

$$u(x', y, z') - u(x', y, z) \geq u(x, y, z') - u(x, y, z). \tag{19}$$

Rewriting each side of (19) as an integral, we get:

$$\int_z^{z'} u_3(x', y, s)ds \geq \int_z^{z'} u_3(x, y, s)ds \tag{20}$$

because (x', y, s) and (x, y, s) belong to the interior of Λ for all $s \in (z, z')$. However, because u_{13} is negative in the interior of Λ, and $x' > x$, we have $u_3(x', y, s) < u_3(x, y, s)$ for each $s \in (z, z')$. This contradicts (20) and establishes that $z' \leq z$.

If (x, y) and (x', y) are in Ω and $x' > x$, then we can approximate (x, y) and (x', y) by $(\tilde{x}, \tilde{y})$ and $(\tilde{x}', \tilde{y})$ in the interior of Ω, with $\tilde{x}' > \tilde{x}$. Then, we have $h(\tilde{x}', \tilde{y}) \leq h(\tilde{x}, \tilde{y})$, and by using the continuity of h on Ω, we obtain $h(x', y) \leq h(x, y)$. □

A tool introduced in Mitra and Nishimura (2005) is the ϕ-policy function defined by:

$$\phi(x) = h(x, x) \textit{ for all } x \in X. \tag{21}$$

This turns out to be useful in the present context too. Clearly, ϕ is a continuous function on X. Using Proposition 2, we can establish the following important single-crossing property of the ϕ-policy function.

Proposition 3 *The ϕ-policy function satisfies*:

$$\left.\begin{array}{ll} (i)\ \phi(x) > x & \textit{for } x \in (0, \bar{x}) \\ (ii)\ \phi(x) < x & \textit{for } x \in (\bar{x}, B] \end{array}\right\}. \tag{22}$$

Proof: We establish (22)(i) as follows. Suppose, on the contrary, there is some $k \in (0, \bar{x})$ for which $\phi(k) \leq k$. If equality holds, then $k = \bar{x}$ by Lemma 1, a contradiction. So we must have $\phi(k) < k$. This, in turn, means that $\phi(x) < x$ for all $x \in (0, \bar{x})$, using the continuity of ϕ.

Denote by (x_t) the optimal program from (k, k). Then, we have:

$$x_2 = h(k, k) = \phi(k) < k \tag{23}$$

and

$$x_3 = h(k, x_2) \leq h(x_2, x_2) = \phi(x_2), \tag{24}$$

the weak inequality in (24) following from (23) and Proposition 2. If $x_2 = 0$, then $x_t = 0$ for $t \geq 2$. If $x_2 > 0$, then $\phi(x_2) < x_2$ and (24) implies that $x_3 < x_2$. Therefore, repeating this argument for all successive t, we get $x_{t+1} \leq x_t$ for $t \geq 0$.

Define

$$x' = (1 - \delta) \sum_{t=0}^{\infty} \delta^t x_{t+1}, \ \ x'' = (1 - \delta) \sum_{t=0}^{\infty} \delta^t x_{t+2}. \tag{25}$$

Therefore, $x_{t+2} \leq k$ for all $t \geq 0$, we must have $x'' \leq k$.

Now, we can write:

$$\begin{aligned} \hat{x} &\equiv (1 - \delta) \sum_{t=0}^{\infty} \delta^t x_t \\ &= (1 - \delta)x_0 + \delta \left[(1 - \delta) \sum_{t=0}^{\infty} \delta^t x_{t+1} \right] \\ &= (1 - \delta)x_0 + \delta x' \\ &= (1 - \delta)x_0 + \delta[(1 - \delta)x_1 + \delta x'']. \end{aligned} \tag{26}$$

Because $(x_t, x_{t+1}, x_{t+2}) \in \Lambda$ for all $t \geq 0$, and

$$(\hat{x}, x', x'') = \left((1-\delta)\sum_{t=0}^{\infty} \delta^t x_t, (1-\delta)\sum_{t=0}^{\infty} \delta^t x_{t+1}, (1-\delta)\sum_{t=0}^{\infty} \delta^t x_{t+2} \right)$$

and $(1-\delta)\sum_{t=0}^{\infty} \delta^t = 1$, and Λ is convex, we have $(\hat{x}, x', x'') \in \Lambda$. Therefore, by Jensen's inequality:

$$\begin{aligned}
&(1-\delta)\sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1}, x_{t+2}) \\
&\leq u\left((1-\delta)\sum_{t=0}^{\infty} \delta^t x_t, (1-\delta)\sum_{t=0}^{\infty} \delta^t x_{t+1}, (1-\delta)\sum_{t=0}^{\infty} \delta^t x_{t+2} \right) \\
&= u(\hat{x}, x', x'') \\
&= u((1-\delta)x_0 + \delta[(1-\delta)x_1 + \delta x''], [(1-\delta)x_1 + \delta x''], x'') \\
&= u((1-\delta^2)k + \delta^2 x'', (1-\delta)k + \delta x'', x'') \\
&\leq u(k, k, k),
\end{aligned} \tag{27}$$

the last inequality in (27) following from Lemma 3, using the fact that $k \in (0, \bar{x})$, and $x'' \leq k$. However, clearly, (27) implies that $(k, k, k, k, \ldots)$ is optimal from k, so that $\phi(k) = k$, a contradiction. This establishes (22)(i).

To establish (22)(ii), we note first that $\phi\ (B) < B$. This follows by using the proof of Proposition 4 in Mitra and Nishimura (2005). Therefore, if (22)(ii) does not hold, there is some $k \in (\bar{x}, B)$ such that $\phi(k) = k$, by continuity of ϕ. However, then, we must have $k = \bar{x}$ by Lemma 1, a contradiction. □

4 Global dynamics

In this section, we analyze the global dynamics of the dynamical system generated by the optimal policy function through the equation (14). It is a two-dimensional dynamical system and, therefore, it is considerably harder to provide a complete picture of the global dynamics in this case, compared to the one-dimensional system generated by the standard one-sector model of Ramsey-optimal growth.

In particular, note that we were concerned in Section 3 with properties of optimal programs starting from $(k, k) \in \Omega$. It is more difficult to study properties of optimal programs starting from an arbitrary $(x, y) \in \Omega$, because this history can affect the utility in the first two periods in ways that are difficult to predict, given our standard assumptions.

Progress with the analysis is considerably simplified if we assume (following Mitra and Nishimura 2005) that the utility function has bounded steepness.

Assumption 8 There is $A > 0$, such that for all (x, y, z), (x', y', z') in Λ, $|u(x, y, z) - u(x', y', z')| \leq A\|(x, y, z) - (x', y', z')\|$.

Assumption 8 ensures that u is Lipschitz-continuous, with Lipschitz constant A. The norm used in Assumption 8 is the sum-norm; that is, $\|(x, y, z)\| = |x| + |y| + |z|$ for (x, y, z) in $\mathbb{R}^3$.

A basic implication of this assumption is noted below.

Lemma 4 *For $(x, y) \in \Omega$ and $(x, y) \gg 0$, we must have $h(x, y) > 0$.*

PROOF: Suppose on the contrary that there is $(x, y) \in \Omega$ with $(x, y) \gg 0$ and $h(x, y) = 0$. Then, $(x_t) \equiv (x, y, 0, 0, 0, \ldots)$ is optimal starting from (x, y). Using Proposition 1, we can find $k \in (0, y)$, such that:

$$V(k, k)/k \geq 3A/\delta^2. \tag{28}$$

Note that $(k, k) \in \Omega$, and let (x'_t) be the optimal program from $(k, k)y$. Using the fact that $k \in (0, y)$, we also have $(y, k) \in \Omega$, and we can define the program:

$$(x''_t) \equiv (x, y, k, k, x'_2, x'_3, \ldots\ldots). \tag{29}$$

Using Assumption 8, we have $u(x, y, 0) - u(x, y, k) \leq Ak$, and $u(y, k, k) - u(y, 0, 0) \leq 2Ak$. Therefore, we obtain:

$$\begin{aligned} &\sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1}, x_{t+2}) - \sum_{t=0}^{\infty} \delta^t u(x''_t, x''_{t+1}, x''_{t+2}) \\ &\quad = u(x, y, 0) + \delta u(y, 0, 0) - \delta^2 V(k, k) \\ &\quad \leq Ak(2 + \delta) - 3Ak < 0, \end{aligned} \tag{30}$$

where we have used the facts that (x'_t) is the optimal program from (k, k), and (28). But, (30) clearly contradicts the fact that (x_t) is optimal starting from (x, y). □

We now state our first result on global dynamics, which might appear to be very weak, but which turns out to be very helpful for the subsequent analysis.

Proposition 4 *If (x_t) is an optimal program from $(x, y) \in \Omega$ with $(x, y) \gg 0$, and $x < \bar{x}$, then the following property cannot hold*:

$$x_{t+1} \leq x_t \text{ for all } t \geq 0. \tag{31}$$

PROOF: Using Lemma 4, we have $x_t > 0$ for $t \geq 0$. Suppose (31) holds. Then, the sequence (x_t) must converge. If it converges to some $m > 0$, then we have $m \in (0, \bar{x})$, and by continuity of h, we must also have $h(m, m) = m$. Therefore, $m = \bar{x}$ by Lemma 1, a contradiction.

This leaves us with the possibility that (x_t) converges to zero. Using Proposition 1, we can find $k' > 0$, such that for all $k \in (0, k')$, we have:

$$V(k, k)/k \geq 4A/\delta(1 - \delta). \tag{32}$$

Because $x_t \to 0$ as $t \to \infty$, there is some $T \geq 2$, such that $x_t \in (0, k')$ for all $t \geq T$. Define $k = x_T$, and note that $(k, k) \in \Omega$. Let (x'_t) be the optimal program from (k, k). Using the fact that $k = x_T$, we can define the program:

$$(x''_t) \equiv (x_0, \ldots, x_{T-1}, k, k, x'_2, x'_3, \ldots\ldots). \tag{33}$$

Using Assumption 8, we have $u(x_{T-1}, x_T, x_{T+1}) - u(x_{T-1}, k, k) \leq Ak$, and $u(x_t, x_{t+1}, x_{t+2}) = u(x_t, x_{t+1}, x_{t+2}) - u(0, 0, 0) \leq 3Ak$ for $t \geq T$, because $x_t \leq x_T = k$ for $t \geq T$ by (31). Therefore, we obtain:

$$\begin{aligned}
&\sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1}, x_{t+2}) - \sum_{t=0}^{\infty} \delta^t u(x''_t, x''_{t+1}, x''_{t+2}) \\
&\quad = \delta^{T-1}[u(x_{T-1}, x_T, x_{T+1}) - u(x_{T-1}, k, k)] \\
&\quad + \sum_{t=T}^{\infty} \delta^t u(x_t, x_{t+1}, x_{t+2}) - \delta^T V(k, k) \\
&\quad \leq \delta^{T-1}[Ak + \{3\delta Ak/(1-\delta)\} - \{4Ak/(1-\delta)\}] < 0,
\end{aligned} \tag{34}$$

where we have used the facts that (x'_t) is the optimal program from (k, k), and (32). But, (34) clearly contradicts the fact that (x_t) is optimal starting from (x, y). □

The single-crossing property of ϕ (Proposition 3) and the monotonicity property of h in the first argument (Proposition 2) lead to a rather strong monotonicity property over time of optimal programs.

Proposition 5 *Let (x_t) be an optimal program from $(x, y) \in \Omega$, with $(x, y) \gg 0$. Then, the following properties hold*:

(i) *If $\bar{x} \geq x_{t+1} \geq x_t$ holds for some t, then $x_{t+2} \geq x_{t+1}$ also holds for that t.*
(ii) *If $\bar{x} \leq x_{t+1} \leq x_t$ holds for some t, then $x_{t+2} \leq x_{t+1}$ also holds for that t.*

PROOF: (i) If $\bar{x} \geq x_{t+1} \geq x_t$ holds for some t, then:

$$x_{t+2} = h(x_t, x_{t+1}) \geq h(x_{t+1}, x_{t+1}) \geq x_{t+1},$$

where the first inequality follows from Proposition 2 and the second inequality follows from Proposition 3.

(ii) If $\bar{x} \leq x_{t+1} \leq x_t$ holds for some t, then:

$$x_{t+2} = h(x_t, x_{t+1}) \leq h(x_{t+1}, x_{t+1}) \leq x_{t+1},$$

where the first inequality follows from Proposition 2 and the second inequality follows from Proposition 3. □

An important corollary of this monotonicity property over time is that if the optimal program is always on the same side (that is, always above or always below) of the nontrivial stationary optimal stock, it must converge to the nontrivial stationary optimal stock.

Corollary 1 *Let (x_t) be an optimal program from $(x, y) \in \Omega$, with $(x, y) \gg 0$. Then, the following properties hold*:

(i) *If $\bar{x} \geq x_t$ holds for all t, then $x_t \to \bar{x}$ as $t \to \infty$.*
(ii) *If $\bar{x} \leq x_t$ holds for all t, then $x_t \to \bar{x}$ as $t \to \infty$.*

PROOF: (i) There are two possibilities to consider. Either (a) $x_{t+1} \leq x_t$ for all $t \geq 0$, or (b) $x_{T+1} > x_T$ for some $T \geq 0$. In case (a), using Proposition 4, we must have $x_t = x_{t+1} = \bar{x}$ for all $t \geq 0$, so (i) holds trivially. In case (b), using Proposition 5, we must have $x_{t+1} \geq x_t$ for all $t \geq T$. Therefore, (x_t) must converge to some $k \leq \bar{x}$. Then continuity of h yields $h(k, k) = k$, and Lemma 1 implies that $k = \bar{x}$, establishing (i).

(ii) There are two possibilities to consider. Either (a) $x_{t+1} \geq x_t$ for all $t \geq 0$, or (b) $x_{T+1} < x_T$ for some $T \geq 0$. In case (a), (x_t) must converge to some $k \in [\bar{x}, B]$. Then continuity of h yields $h(k, k) = k$, and Proposition 3 implies that $k < B$. Now, Lemma 1 can be used to conclude that $k = \bar{x}$. So, in this case, we actually have $x_t = x_{t+1} = \bar{x}$ for all $t \geq 0$, and (i) holds trivially. In case (b), using Proposition 5, we must have $x_{t+1} \leq x_t$ for all $t \geq T$. Therefore, (x_t) must converge to some $k \geq \bar{x}$. Then continuity of h yields $h(k, k) = k$, and Lemma 1 implies that $k = \bar{x}$, establishing (ii). □

This result should be compared to the turnpike property in the standard one-sector model of Ramsey-optimal growth, where optimal programs always lie on the same side of the nontrivial stationary optimal stock: it is not optimal to cross-over the stationary optimal stock. In the current context, given that the relevant dynamical system is two-dimensional, such cross-overs cannot be ruled out by optimality arguments.

We can now summarize the global dynamics of optimal programs in the following statement.

Theorem 1 *Let (x_t) be an optimal program from $(x, y) \in \Omega$, with $(x, y) \gg 0$. Then, (x_t) satisfies one of the following two properties*:

$$\left.\begin{array}{l} (a)\lim_{t\to\infty} x_t = \bar{x} \\ (b)\limsup_{t\to\infty} x_t > \liminf_{t\to\infty} x_t \\ and\ \bar{x} \in [\liminf_{t\to\infty} x_t, \limsup_{t\to\infty} x_t] \end{array}\right\}. \tag{35}$$

Case (a) always holds if $x_t \leq \bar{x}$ for all $t \geq 0$, or if $x_t \geq \bar{x}$ for all $t \geq 0$.

PROOF: We have three possibilities to consider: (i) there is some $T \geq 0$, such that $x_t \leq \bar{x}$ for all $t \geq T$; (ii) there is some $S \geq 0$, such that $x_t \geq \bar{x}$ for all $t \geq S$; (iii) there is an infinite number of time periods t for which $x_t > \bar{x}$, and there is an infinite number of time periods t for which $x_t < \bar{x}$.

In case (i), using Lemma 4 and Corollary 1, we have $\lim_{t\to\infty} x_t = \bar{x}$, so (35) (a) holds. In case (ii), by Corollary 1, we have $\lim_{t\to\infty} x_t = \bar{x}$, so (35) (a) holds. In case (iii), we have $\limsup_{t\to\infty} x_t \geq \bar{x}$, and $\liminf_{t\to\infty} x_t \leq \bar{x}$. If $\limsup_{t\to\infty} x_t = \liminf_{t\to\infty} x_t$ then, again, (35)(a) holds. If $\limsup_{t\to\infty} x_t > \liminf_{t\to\infty} x_t$, then (35) (b) holds. □

5 Persistent fluctuations

The global dynamical properties obtained in Theorem 1 give a useful classification of optimal dynamic behavior. Specifically, they suggest a way to investigate conditions on the model economy that lead to "persistent fluctuations"; that is, fluctuations that never peter out completely.

Specifically, Theorem 1 indicates that if one can identify conditions under which the global asymptotic stability scenario of possibility (a) can be ruled out, then possibility (b) must occur, leading to persistent fluctuations.

Because possibility (a) makes the unique (nontrivial) stationary stock a global attractor of the dynamical system (14), possibility (a) will be negated if the stationary optimal stock is locally a repellor. This suggests that one might be able to propose purely local conditions that hold near the non-trivial stationary optimal stock to ensure the global result of possibility (b) of Theorem 1.[4]

5.1 Local dynamics

In this subsection, we investigate carefully the local dynamics of optimal solutions near the nontrivial stationary optimal stock. We do this by examining the characteristic roots associated with the fourth-order difference equation, which represents the linearized version of solutions to the Ramsey–Euler equations near the nontrivial stationary optimal stock.

Consider the Ramsey–Euler equation:

$$u_3(x_t, x_{t+1}, x_{t+2}) + \delta u_2(x_{t+1}, x_{t+2}, x_{t+3}) + \delta^2 u_1(x_{t+2}, x_{t+3}, x_{t+4}) = 0. \tag{36}$$

In particular, of course, $x_{t+s} = x^*$ for $s = 0, 1, 2, 3, 4$ satisfies (36):

$$u_3(x^*, x^*, x^*) + \delta u_2(x^*, x^*, x^*) + \delta^2 u_1(x^*, x^*, x^*) = 0. \tag{37}$$

The characteristic equation associated with the Ramsey–Euler equation (36) is given by:

$$\delta^2 u_{13}\beta^4 + \left(\delta^2 u_{12} + \delta u_{23}\right)\beta^3 + \left(\delta^2 u_{11} + \delta u_{22} + u_{33}\right)\beta^2 + (\delta u_{21} + u_{32})\beta + u_{31} = 0. \tag{38}$$

In what follows we drop the points of evaluation of derivatives, it being understood that the point of evaluation for derivatives of the production function, f, is $\bar{x}$, the point of evaluation for derivatives of the welfare function, w, is $(\bar{c}, \bar{c})$, where $\bar{c} = f(\bar{x}) - \bar{x}$, and the point of evaluation for derivatives of the utility function, u, is $(\bar{x}, \bar{x}, \bar{x})$.

It is convenient for our analysis to define the following magnitudes:

$$a = \frac{\delta(-w_{11}) + (-w_{22})}{\sqrt{\delta}\, w_{12}}, \tag{39}$$

$$b = (1/\sqrt{\delta}) + \sqrt{\delta}, \tag{40}$$

$$C = \frac{\delta(-f''(k))[\delta w_1 + w_2]}{w_{12}} \tag{41}$$

[4] The idea of using local conditions at a stationary state to produce persistent fluctuations globally in two-dimensional dynamical systems is contained in a paper by Sedaghat (1998). However, the dynamical system considered by him is not obtained from an optimization model.

and

$$\gamma \equiv \{(1+\delta)w_{12} - [\delta(-w_{11}) + (-w_{22})]\}. \tag{42}$$

Notice that $\beta = 0$ is not a solution to (38) because $u_{13} \neq 0$. We can, therefore, use the transformed variable,

$$\mu = \delta\beta + (1/\beta),$$

to examine the roots of (38). Using this transformation, (38) becomes:

$$u_{13}\mu^2 + (\delta u_{12} + u_{23})\mu + \left[\delta^2 u_{11} + \delta u_{22} + u_{33} - 2\delta u_{13}\right] = 0. \tag{43}$$

Denoting the roots of (43), which is a quadratic in μ, by μ_1 and μ_2, we note that if:

$$\delta^2 < \frac{\gamma^2}{4(-f''(k))w_{12}[\delta w_1 + w_2]} \tag{C}$$

then these roots are real and positive. Condition (C) requires that the discount factor be small relative to an expression involving the curvature of the welfare and production functions.[5]

Given $\mu_i (i = 1, 2)$, we can obtain the corresponding roots of β by solving the quadratic:

$$\delta\beta + (1/\beta) = \mu_i. \tag{44}$$

We denote the roots of (44) corresponding to μ_1 by β_1 and β_2 (with $|\beta_1| = \min[|\beta_1|,\ |\beta_2|]$) and the roots of (44) corresponding to μ_2 by β_3 and β_4 (with $|\beta_3| = \min[|\beta_3|,\ |\beta_4|]$). If we have strong intertemporal dependence so that:

$$\gamma \equiv \{(1+\delta)w_{12} - [\delta(-w_{11}) + (-w_{22})]\} > 0, \tag{SD}$$

then β_j (for $j = 1, 2, 3, 4$) are real and $\beta_j > 1$ for $j = 1, 2, 3, 4$. Note that (SD) is consistent with having the Hessian of w negative-definite at the steady state. We can now state the following result.

Proposition 6 *Under the conditions (C) and (SD), the roots of the characteristic equation (38) associated with the Ramsey–Euler equation satisfy*:

$$1 < \beta_1 < 1/\sqrt{\delta} < \beta_2;\ 1 < \beta_3 < 1/\sqrt{\delta} < \beta_4. \tag{45}$$

PROOF: We break up the proof into several steps.

Step 1 (Condition [C] is equivalent to $4C < (b-a)^2$).

[5] For the class of dynamic optimization problems described by (5), a sufficient condition for global asymptotic stability is that the discount factor be large enough relative to an expression involving the curvature of the reduced form utility function. For this result, see especially Cass and Shell (1976), Brock and Scheinkman (1978), and for a recent exposition, Mitra (2005). In view of this, it might not be totally unexpected that a condition like (C), (together with condition [SD]) is used to ensure persistent fluctuations. However, it should be remembered that (C) is a purely local condition, unlike the conditions used to ensure global asymptotic stability.

We have:

$$(b - a) = \frac{(1+\delta)}{\sqrt{\delta}} - \frac{[\delta(-w_{11}) + (-w_{22})]}{w_{12}\sqrt{\delta}} = \frac{(1+\delta)w_{12} - [\delta(-w_{11}) + (-w_{22})]}{w_{12}\sqrt{\delta}}$$

so that:

$$(b-a)^2 = \frac{\{(1+\delta)w_{12} - [\delta(-w_{11}) + (-w_{22})]\}^2}{\delta(w_{12})^2} = \frac{\gamma^2}{\delta(w_{12})^2}.$$

Therefore, the condition $4C < (b-a)^2$ can be written as:

$$\frac{4\delta(-f''(k))[\delta w_1 + w_2]}{w_{12}} < \frac{\gamma^2}{\delta(w_{12})^2}.$$

Because $w_{12} > 0$, this is clearly equivalent to condition (C).

Step 2 (The condition $4C < (b-a)^2$ implies that μ_1, μ_2 are real).
Define:

$$C' = \frac{\left[\delta^2 u_{11} + \delta u_{22} + u_{33} - 2\delta u_{13}\right]}{u_{13}}; b' = \frac{-(\delta u_{12} + u_{23})}{u_{13}} > 0.$$

Using (43), to show that μ_1, μ_2 are real, we need to check that $(b')^2 > 4C'$. To this end, note that:

$$C' = \frac{(-w_{11})\delta(1+\delta) + (-w_{22})(1+\delta) + \delta^2(-f'')[\delta w_1 + w_2]}{w_{12}} > 0$$

and:

$$\frac{(b')^2}{\delta} = \frac{\{-(\delta u_{12} + u_{23})\}^2}{\delta(u_{13})^2} = \frac{\{[\delta(-w_{11}) + (-w_{22})] + (1+\delta)w_{12}\}^2}{\delta(w_{12})^2} = (a+b)^2.$$

Because we have:

$$\delta(a+b)^2 - \delta(a-b)^2 = 4\delta ab = 4a(1+\delta)\sqrt{\delta}$$

we obtain:

$$(b')^2 = \delta(a+b)^2 = \delta(a-b)^2 + 4a(1+\delta)\sqrt{\delta} > 4\delta c + 4a(1+\delta)\sqrt{\delta} = 4[\delta c + a(1+\delta)\sqrt{\delta}], \quad (46)$$

where we have used $4C < (b - a)^2$ to obtain the strict inequality in (46). Using the definitions of a and C, we have:

$$\begin{aligned} a(1+\delta)\sqrt{\delta} + \delta C &= \frac{[\delta(1+\delta)(-w_{11}) + (1+\delta)(-w_{22})]}{w_{12}} + \delta C \\ &= \frac{[\delta(1+\delta)(-w_{11}) + (1+\delta)(-w_{22})] + \delta^2(-f'')(\delta w_1 + w_2)}{w_{12}} \\ &= C'. \end{aligned}$$

Using this information in (46) yields $(b')^2 > 4C'$, as required.

Step 3 (Condition [SD] is equivalent to $b > a$).

Using the definitions of a and b, the condition $b > a$ can be written as:

$$\frac{(1+\delta)}{\sqrt{\delta}} > \frac{[\delta(-w_{11}) + (-w_{22})]}{w_{12}\sqrt{\delta}},$$

which is equivalent to:

$$(1+\delta)w_{12} > [\delta(-w_{11}) + (-w_{22}),$$

and this is condition (SD).

Step 4 (Condition $b > a$ implies that $\mu_i < (1+\delta)$ for $i = 1, 2$).

Suppose for some $i \in \{1, 2\}$, we have $\mu_i \geq (1+\delta)$. Therefore, we have:

$$\frac{b' \pm \sqrt{(b')^2 - 4C'}}{2} \geq (1+\delta). \tag{47}$$

If (47) holds with the minus sign, then:

$$b' \geq 2(1+\delta) + \sqrt{(b')^2 - 4C'} > 2(1+\delta)$$

by Step 2. However, by the calculations of Step 2 above, $b' = \sqrt{\delta}(a+b) < 2b\sqrt{\delta}$, by using $a < b$. Therefore, we must have $b' < 2(1+\delta)$, and (47) must hold with the plus sign. That is,

$$\sqrt{(b')^2 - 4C'} \geq 2(1+\delta) - b'. \tag{48}$$

Now, by the calculations of Step 2, $b' = a\sqrt{\delta} + b\sqrt{\delta} = a\sqrt{\delta} + (1+\delta)$, so that:

$$\begin{aligned} 2(1+\delta) - b' &= (1+\delta) - a\sqrt{\delta} \\ &= \sqrt{\delta}[(1/\sqrt{\delta}) + \sqrt{\delta} - a] \\ &= \sqrt{\delta}(b-a). \end{aligned}$$

Using this in (48), we get:

$$(b')^2 - 4C' \geq \delta(b-a)^2. \tag{49}$$

By the calculations of Step 2, we have:

$$\begin{aligned}(b')^2 - 4C' &= \delta(a-b)^2 + 4a(1+\delta)\sqrt{\delta} - [4a(1+\delta)\sqrt{\delta} + 4\delta C] \\ &= \delta(a-b)^2 - 4\delta C.\end{aligned} \tag{50}$$

Using (49) and (50), we get:

$$\delta[(a-b)^2 - 4C] \geq \delta(b-a)^2.$$

However, this is clearly a contradiction, because $C > 0$.

Step 5 (Condition $a > 2$ holds, using $w_{12} > 0$).

The condition $a > 2$ is equivalent to:

$$\frac{[\delta(-w_{11}) + (-w_{22})]}{w_{12}\sqrt{\delta}} > 2. \tag{51}$$

Because the Hessian of w is negative definite, we have:

$$\begin{aligned}&[\sqrt{\delta}\ 1]\begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22}\end{bmatrix}\begin{bmatrix}\sqrt{\delta} \\ 1\end{bmatrix} \\ &= \delta w_{11} + 2w_{12}\sqrt{\delta} + w_{22} < 0.\end{aligned}$$

and this is clearly equivalent to (51), because $w_{12} > 0$.

Step 6 (Condition $a > 2$ implies $\mu_i > 2\sqrt{\delta}$ for $i = 1, 2$).

Suppose $\mu_i \leq 2\sqrt{\delta}$ for some $i \in \{1, 2\}$. Because (by Step 4), we have $\mu_j < (1+\delta)$ for $j \neq i$, we obtain:

$$\mu_1 + \mu_2 < (1+\delta) + 2\sqrt{\delta} = (1+\sqrt{\delta})^2. \tag{52}$$

However, $\mu_1 + \mu_2 = b'$, and by the calculations of Step 2, and condition $a > 2$, we have:

$$\begin{aligned}b' &= \sqrt{\delta}(a+b) \\ &> 2\sqrt{\delta} + (1+\delta) = (1+\sqrt{\delta})^2,\end{aligned}$$

which contradicts (52).

Step 7 ($\mu_i > 2\sqrt{\delta}$ for $i = 1, 2$ implies that β_j are real for $j \in \{1, 2, 3, 4\}$).

Since $\mu_1 > 2\sqrt{\delta}$, we have $(\mu_1)^2 > 4\delta$, and so the roots β_1 and β_2, given by:

$$\beta_j = \frac{\mu_1 \pm \sqrt{(\mu_1)^2 - 4\delta}}{2}$$

are clearly real. The same argument, using $\mu_2 > 2\sqrt{\delta}$, can be used to show that β_3 and β_4 are real.

Step 8 ($\beta_j > 1$ for $j \in \{1, 2, 3, 4\}$).

We have $\beta_1\beta_2 = (1/\delta) > 0$ and $(\beta_1 + \beta_2) = \mu_1/\delta > 0$, so β_1 and β_2 must be positive. Suppose $\beta_1 \leq 1$. We have $\beta_2 = (1/\delta\beta_1)$, so that $(\beta_1 + \beta_2) = \beta_1 + (1/\delta\beta_1)$.

Define:

$$g(x) = x + (1/\delta x) \text{ for } x > 0.$$

Then, we have $g'(x) = 1 - (1/\delta x^2) < 0$ for $x \in (0, 1]$. Therefore, g attains a minimum on $(0, 1]$ at $x = 1$, and so $g(x) \geq g(1) = 1 + (1/\delta)$.

Because $\beta_1 \leq 1$, we have $(\beta_1 + \beta_2) = g(\beta_1) \geq 1 + (1/\delta)$. Therefore, we obtain:

$$\mu_1/\delta = (\beta_1 + \beta_2) \geq (1 + \delta)/\delta$$

so that $\mu_1 \geq (1 + \delta)$, which contradicts Step 4. Thus $\beta_1 > 1$, and so $\beta_2 > 1$. Similarly, one establishes that $\beta_3 > 1$ and $\beta_4 > 1$.

Step 9 (Establishing [45]).

Because $\mu_1 > 2\sqrt{\delta}$, we have $(\mu_1)^2 - 4\delta > 0$, and so $\beta_2 > \beta_1 > 1$. Because $\beta_1\beta_2 = (1/\delta)$, we must have $\beta_2 > (1/\sqrt{\delta}) > \beta_1 > 1$. By the same reasoning, $\beta_4 > (1/\sqrt{\delta}) > \beta_3 > 1$. □

Remark Our local analysis parallels that offered in Boyer (1978), although our method, using the transformation (44), is slightly different from his. Because the Hessian of w at the steady state is negative definite, we have:

$$\delta(-w_{11}) + (-w_{22}) > 2\, w_{12}\sqrt{\delta}.$$

Because $w_{12} > 0$, this ensures that $a > 2$, as checked in Step 5 of the proof of Proposition 6. Boyer assumes $a > 2$ separately. The curvature condition (C) corresponds to Boyer's condition:

$$4C < (b - a)^2. \tag{53}$$

However, Boyer does not define C in terms of the primitives. If C is defined as in (41), then (53) is equivalent to condition (C), as checked in Step 2 of the proof of Proposition 6. The strong dependence condition (SD) corresponds to Boyer's condition:

$$b > a,$$

as checked in Step 3 of the proof of Proposition 6.

5.2 An example

One might get the impression from the above analysis that the local instability scenario imposed by conditions (C) and (SD) arise when the discount factor is sufficiently low. Actually, conditions (C) and (SD), which ensure that the stationary optimal stock $\bar{x}$ is locally a repellor, do not impose uniform discount factor restrictions (independent of w and f). To elaborate, given any discount factor $\delta \in (0, 1)$, one can construct (w, f) satisfying all the standard assumptions, such that conditions (C) and (SD) are met at the unique steady state.

Let $\delta \in (0, 1)$ be given. Denote $\lambda = \sqrt{\delta}$, and:

$$H(\lambda) = \frac{(1-\lambda)^4\lambda}{16(1+\lambda)(2+\lambda^2)^2}. \tag{54}$$

Now, define:

$$B' = \min\{1/4\delta, 1/2, H(\lambda)\}$$
$$A' = (1/\delta) + 2B'.$$

We now proceed to define the production function as follows. Let:

$$f(x) = \begin{cases} A'x - B'x^2 & \text{for } x \in [0, 2] \\ f(2) + \frac{A''(x-2)}{[B''+(x-2)]} & \text{for } x > 2 \end{cases}, \tag{55}$$

where:

$$B'' = \{(1/\delta) - 2B'\}/B'$$
$$A'' = \{(1/\delta) - 2B'\}^2/B'.$$

Note that $B'' = (1/\delta B') - 2 \geq 4 - 2 = 2$, by definition of B'. Furthermore, we have

$$\begin{aligned} A'' &= \frac{\{(1/\delta) - 2B'\}}{B'}\{(1/\delta) - 2B'\} \\ &= B''\{(1/\delta) - 2B'\} < B''/\delta. \end{aligned}$$

It can be checked that $f(0) = 0$ and f is increasing, strictly concave and twice continuously differentiable on $\mathbb{R}_+$, with $f'(0) = A' > 1$ and $f'(x) \to 0$ as $x \to \infty$.

Furthermore, for $\bar{B} = (B'/\delta)$, one has $\bar{B} > 2$ and so:

$$f(\bar{B}) = \frac{A''\bar{B}}{B'' + (\bar{B} - 2)} < \frac{(B''/\delta)}{B'' + (\bar{B} - 2)}\bar{B}. \tag{56}$$

It can be checked that $[(\bar{B} - 2) + B''] \geq (B''/\delta)$, and using this information in (56), we see that $f(\bar{B}) < \bar{B}$; therefore, denoting the maximum sustainable stock by B, we have $B < \bar{B}$.

For the production function, f, we have:

$$\delta f'(1) = \delta[A' - 2B'] = 1,$$

so that the unique nontrivial stationary optimal stock, $\bar{x} = 1$. Furthermore, $f(\bar{x}) = A' - B'$, and $\bar{c} = A' - B' - 1$. Note that:

$$\bar{c} = (1/\delta) + 2B' - B' - 1 = (1/\delta) + B' - 1 < (1/\delta). \tag{57}$$

Now, we proceed to define the welfare function, w, as follows. Denote $\beta = (\delta/2)$, $\alpha = \mu = 1$, $\nu = \beta\lambda$, and $\theta = 4\beta\lambda/(1+\lambda)$, and define:

$$w(c, d) = \alpha c - \beta c^2 + \mu d - \nu d^2 + \theta cd \text{ for } (c, d) \in K \times K,$$

where $K = [0, (1/\delta)]$. Clearly, w is twice continuously differentiable on K^2, and we can calculate:

$$\begin{aligned} w_1(c, d) &= \alpha - 2\beta c + \theta d > \alpha - 2\beta(1/\delta) = 0 \\ w_2(c, d) &= \mu - 2\nu d + \theta c > \mu - 2\nu(1/\delta) > 0 \\ w_{11}(c, d) &= -2\beta < 0 \\ w_{22}(c, d) &= -2\nu < 0 \\ w_{12}(c, d) &= w_{21}(c, d) = \theta > 0. \end{aligned} \tag{58}$$

To show that the Hessian of w is negative definite, we need to check that $\theta^2 < 4\beta\nu$. By definition of θ, we have:

$$\begin{aligned} \theta^2 &= 16\beta^2\lambda^2/(1+\lambda)^2 \\ &= (4\beta\lambda)(4\nu)/(1+\lambda)^2 \\ &= (4\nu\beta)[4\lambda/(1+\lambda)^2] \\ &< 4\nu\beta, \end{aligned}$$

because $(1+\lambda)^2 = (1-\lambda)^2 + 4\lambda > 4\lambda$ for $\lambda \in (0, 1)$. Therefore, the assumption (w) on w is satisfied on $K \times K$.

Remark Note that we only define w on K^2, and show that the assumption (w) on w is satisfied on K^2. Now, w can be extended to $\mathbb{R}^2_+$, preserving all these properties; this is a tedious step that we do not explicitly present. Note also that because the relevant local conditions (SD) and (C) will need to be checked only at $(\bar{c}, \bar{c})$, which belongs to K^2 by (57), it is only the behavior of w on K^2 that is of significance in what follows.

5.2.1 Checking Condition SD

We now proceed to check that condition SD is satisfied. Define:

$$\phi(z) = 4\beta z/(1+z) \text{ for } z \geq 0.$$

Then, we have $\phi'(z) = 4\beta/(1+z)^2 > 0$. Therefore, we have:

$$\theta = 4\beta\lambda/(1+\lambda) > 4\beta\delta/(1+\delta)$$

because $\lambda = \sqrt{\delta} > \delta$. We can write this as:

$$\theta > [2\beta\delta/(1+\delta)] + [2\nu/(1+\delta)].$$

Using (58), we then have for every $(c, d) \in K^2$:

$$w_{12}(c, d) > (-w_{11}(c, d)[\delta/(1+\delta)] + (-w_{22}(c, d))[1/(1+\delta)]$$

and this verifies condition (SD).

Remark Because $w_{12}(c, d) = \theta > 0$ for $(c, d) \in K^2$, and condition (SD) holds, we have:

$$a = \frac{\delta(-w_{11}(c, d)) + (-w_{22}(c, d))}{w_{12}(c, d)\sqrt{\delta}} > 2$$

for all $(c, d) \in K^2$, following Step 5 of the proof of Proposition 6.

5.2.2 Checking Condition C

We now proceed to check that Condition (C) is satisfied. As already shown in Step 1 of the proof of Proposition 6, this is equivalent to checking:

$$4C < (b-a)^2, \tag{59}$$

where a, b, C are as defined in (39), (40) and (41), respectively.

For our example, we have (dropping the point of evaluation $(\bar{c}, \bar{c})$ of the derivatives):

$$\begin{aligned}(b-a) &= \frac{(1+\delta)w_{12} - [\delta(-w_{11}) + (-w_{22})]}{w_{12}\sqrt{\delta}} \\ &= \frac{(1+\delta)\theta - [2\delta\beta + 2\nu]}{\theta\sqrt{\delta}} \\ &= \frac{(1+\lambda^2)}{\lambda} - \frac{2\lambda\beta[1+\lambda]}{\theta\lambda} \\ &= \frac{(1+\lambda^2)}{\lambda} - \frac{2\beta[1+\lambda]^2}{4\beta\lambda} \\ &= \frac{(1+\lambda^2)}{\lambda} - \frac{[1+\lambda]^2}{2\lambda} = \frac{[1-\lambda]^2}{2\lambda},\end{aligned}$$

so that:

$$(b-a)^2 = \frac{(1-\lambda)^4}{4\lambda^2}. \tag{60}$$

Furthermore, for our example,

$$4C = (8\delta B'/\theta)[\delta w_1 + w_2].$$

We now observe, using (57) and (58), that:

$$\delta w_1 \leq \delta(1+\theta\bar{c}) < \delta(1+(\theta/\delta)) = (\delta+\theta) \leq (2+\delta) \tag{61}$$

because $\theta \leq 4\beta = 2\delta < 2$. Similarly, using (57) and (58),

$$w_2 \leq (1+\theta\bar{c}) < (1+(\theta/\delta)) < (2+\delta)/\delta. \tag{62}$$

Combining (61) and (62),

$$\delta w_1 + w_2 < (2+\delta)(1+\delta)/\delta < (2+\delta)^2/\delta. \tag{63}$$

Using (63), we obtain:

$$\begin{aligned} 4C &< (8B'/\theta)(2+\delta)^2 \\ &= \frac{8B'(1+\lambda)(2+\lambda^2)^2}{4\lambda\beta} \\ &= \gamma\frac{16B'(1+\lambda)(2+\lambda^2)^2}{4\lambda\delta} \\ &\le \frac{16(1+\lambda)(2+\lambda^2)^2}{4\lambda\delta}\,\frac{(1-\lambda)^4\lambda}{16(1+\lambda)(2+\lambda^2)^2} \\ &= \frac{(1-\lambda)^4}{4\lambda^2} \end{aligned} \tag{64}$$

using the definition of B'. Using (60) and (64), condition (C) is verified.

5.3 The main result

Using Theorem 1 and Proposition 6, we obtain the following global result, based on local conditions at the steady state. It states that every optimal program (except the nontrivial stationary optimal program) must exhibit persistent fluctuations; that is, fluctuations that do not dampen over time.

Theorem 2 *Let* (x_t) *be an optimal program from* $(x, y) \in \Omega$, *with* $(x, y) \gg 0$, *and* $(x, y) \neq (\bar{x}, \bar{x})$. *Under conditions* (*C*) *and* (*SD*), (x_t) *satisfies the following property*:

$$\limsup_{t\to\infty} x_t > \liminf_{t\to\infty} x_t \text{ and } \bar{x} \in \left[\liminf_{t\to\infty} x_t, \limsup_{t\to\infty} x_t\right]. \tag{65}$$

Proof: The proof involves several steps. First, we define a (four dimensional) map that represents solutions to the Ramsey–Euler equations from an arbitrary initial point; this is done by applying the implicit function theorem. Second, we use this map to define a diffeomorphism. Third, we apply the Hartman–Grobman theorem to this diffeomorphism to obtain a homeomorphism that makes the nonlinear map topologically conjugate to the linear map obtained by evaluating the Jacobian matrix of the nonlinear map at its stationary solution. Fourth, we use the fact that all the characteristic roots of this matrix exceed unity to infer that an optimal path from positive initial conditions cannot converge to the nontrivial stationary optimal stock. One then obtains the conclusion of Theorem 2 by applying Theorem 1.

Step 1 Because $f(\bar{x}) > \bar{x}$, we can choose $\eta_1 \in (0, \bar{x})$ such that $f(\bar{x} - \eta_1) > \bar{x} + \eta_1$. Denote the open interval $(\bar{x} - \eta_1, \bar{x} + \eta_1)$ by I_1, and note that $I_1^2 \subset int\,\Omega$. Define $L : I_1^5 \to \mathbb{R}$ by:

$$L(z_1, z_2, z_3, z_4, z_5) = u_3(z_1, z_2, z_3) + \delta u_2(z_2, z_3, z_4) + \delta^2 u_1(z_3, z_4, z_5).$$

Because $(z_1, z_2, z_3) \in int\,\Lambda$, $(z_2, z_3, z_4) \in int\,\Lambda$, $(z_3, z_4, z_5) \in int\,\Lambda$, the map L is well defined and is a C^1 function on its domain. Note that $L(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x}) = 0$ because $\bar{x}$

is a stationary optimal stock with $(\bar{x}, \bar{x}, \bar{x}) \in int\Lambda$, so that it satisfies the Ramsey–Euler equations. Because $u_{13}(\bar{x}, \bar{x}, \bar{x}) \neq 0$ by assumption (w), we can use the implicit function theorem to obtain a neighborhood $U' \subset I_1^4$ of $(\bar{x}, \bar{x}, \bar{x}, \bar{x})$ and a neighborhood $W' \subset I_1$ of $\bar{x}$ and a unique function $M : U' \to W'$ such that:

(i) $L(z_1, z_2, z_3, z_4, M(z_1, z_2, z_3, z_4)) = 0$ for all $(z_1, z_2, z_3, z_4) \in U'$; and
(ii) $M(\bar{x}, \bar{x}, \bar{x}, \bar{x}) = \bar{x}$.

Furthermore, M is a C^1 function on U'. Using (i), we have:

$$u_{13}(\bar{x}, \bar{x}, \bar{x}) + \delta^2 u_{13}(\bar{x}, \bar{x}, \bar{x}) M_1(\bar{x}, \bar{x}, \bar{x}, \bar{x}) = 0$$

so that $M_1(\bar{x}, \bar{x}, \bar{x}, \bar{x}) = -(1/\delta^2) < 0$, because $u_{13}(\bar{x}, \bar{x}, \bar{x}) \neq 0$. Define $G : U' \to \mathbb{R}^4$ by:

$$\begin{aligned} G^1(z_1, z_2, z_3, z_4) &= z_2 \\ G^2(z_1, z_2, z_3, z_4) &= z_3 \\ G^3(z_1, z_2, z_3, z_4) &= z_4 \\ G^4(z_1, z_2, z_3, z_4) &= M(z_1, z_2, z_3, z_4). \end{aligned} \tag{66}$$

Step 2 Clearly, G is a C^1 function on U', and its Jacobian at $(\bar{x}, \bar{x}, \bar{x}, \bar{x})$ is $-M_1(\bar{x}, \bar{x}, \bar{x}, \bar{x}) \neq 0$; furthermore, $G(\bar{x}, \bar{x}, \bar{x}, \bar{x}) = (\bar{x}, \bar{x}, \bar{x}, \bar{x})$. Therefore, by the inverse function theorem,

(i) there exist open sets U and W in $\mathbb{R}^4$ such that $(\bar{x}, \bar{x}, \bar{x}, \bar{x}) \in U$, $(\bar{x}, \bar{x}, \bar{x}, \bar{x}) \in W$, G is one-to-one on U and $G(U) = W$; and
(ii) there is a unique map $g : W \to U$, which satisfies $g(G(z)) = z$ for all $z \in U$.

Furthermore, g is C^1 on W.

Therefore, $G : U \to W$ is a diffeomorphism.

Step 3 The characteristic values of the Jacobian matrix of G at $(\bar{x}, \bar{x}, \bar{x}, \bar{x})$ are given by β_i for $i = 1, 2, 3, 4$, where the β_i refer to the roots obtained from equation (44). This demonstration follows Mitra and Nishimura (2005) and is omitted. Because $\beta_i > 1$ for $i = 1, 2, 3, 4$ under conditions (C) and (SD), $(\bar{x}, \bar{x}, \bar{x}, \bar{x})$ is a hyperbolic fixed point of G.

Applying the Hartman–Grobman theorem (see Hartman 1964; Nitecki 1971), there is a neighborhood $\mathbb{U}'$ of $(\bar{x}, \bar{x}, \bar{x}, \bar{x})$ and a homeomorphism H on $\mathbb{U}'$, such that:

$$H(G(z)) = AH(z) \text{ for all } z \in \mathbb{U}', \tag{67}$$

where A is the Jacobian matrix of G at $(\bar{x}, \bar{x}, \bar{x}, \bar{x})$. Because $G(\bar{x}, \bar{x}, \bar{x}, \bar{x}) = (\bar{x}, \bar{x}, \bar{x}, \bar{x})$, and $(\bar{x}, \bar{x}, \bar{x}, \bar{x})$ is a hyperbolic fixed point of G, (67) implies that $H(\bar{x}, \bar{x}, \bar{x}, \bar{x}) = (0, 0, 0, 0)$.

Clearly, we can find $0 < \varepsilon < \eta_1$, such that $\mathbb{U}^4 \subset \mathbb{U}'$, where $\mathbb{U} = [\bar{x} - \varepsilon, \bar{x} + \varepsilon]$. Therefore, we have:

$$H(G(z)) = AH(z) \text{ for all } z \in \mathbb{U}^4. \tag{68}$$

Step 4 Suppose that (x_t) is an optimal program from some $(x, y) \gg 0$, such that $(x, y) \neq (\bar{x}, \bar{x})$, and:

$$\lim_{t \to \infty} x_t = \bar{x}.$$

Then, there is T, such that:

$$x_t \in \mathbb{U} \text{ for all } t \geq T. \tag{69}$$

There are two possibilities to consider: (A) there is τ such that $x_t = \bar{x}$ for all $t \geq \tau$; (B) there is an infinite number of time periods t, for which $x_t \neq \bar{x}$.

In case (A), we consider the smallest τ for which $x_t = \bar{x}$ for all $t \geq \tau$. Then, $x_{\tau-1} \neq \bar{x}$, and we have, by the optimality of the program (x_t), $h(x_{\tau-1}, \bar{x}) = \bar{x}$. Because we also know that $h(\bar{x}, \bar{x}) = \bar{x}$, it follows from Proposition 2 that for every $\lambda \in (0, 1)$, we must have $h(\lambda x_{\tau-1} + (1-\lambda)\bar{x}, \bar{x}) = \bar{x}$. Choose $\lambda' \in (0, 1)$ close enough to 0 to ensure that $(\lambda' x_{\tau-1} + (1-\lambda')\bar{x}, \bar{x}) \in int\, \Omega$, and define $x' = \lambda' x_{\tau-1} + (1-\lambda')\bar{x}$. Then, $(x', \bar{x}, \bar{x}, \bar{x}, \ldots)$ is optimal from x', and so the following Ramsey–Euler equation holds:

$$u_3(x', \bar{x}, \bar{x}) + \delta u_2(\bar{x}, \bar{x}, \bar{x}) + \delta^2 u_1(\bar{x}, \bar{x}, \bar{x}) = 0.$$

Also, because $\bar{x}$ is a stationary optimal stock with $(\bar{x}, \bar{x}, \bar{x}) \in int\, \Lambda$, we have:

$$u_3(\bar{x}, \bar{x}, \bar{x}) + \delta u_2(\bar{x}, \bar{x}, \bar{x}) + \delta^2 u_1(\bar{x}, \bar{x}, \bar{x}) = 0.$$

However, because $u_{13} \neq 0$, the above two equations cannot hold simultaneously. Therefore, case (A) cannot occur.

So, case (B) must occur. In this case, we can choose $N \geq T$ such that $x_N \neq \bar{x}$, and using (68) and (69), we get:

$$H(G(z_t)) = AH(z_t) \text{ for all } t \geq T, \tag{70}$$

where $z_t = (x_t, x_{t+1}, x_{t+2}, x_{t+3})$ for $t \geq T$. In particular, this gives us:

$$H(z_{N+s}) = A^s H(z_N) \text{ for } s = 1, 2, \ldots \tag{71}$$

Because $x_N \neq \bar{x}$, we have $z_N \neq (\bar{x}, \bar{x}, \bar{x}, \bar{x})$, and because $H(\bar{x}, \bar{x}, \bar{x}, \bar{x}) = (0, 0, 0, 0)$; we know that $H(z_N) \neq (0, 0, 0, 0)$, otherwise H would not be a homeomorphism.

We denote the inverse of A by P, and note that the characteristic roots of P are $(1/\beta_i)$ for $i = 1, 2, 3, 4$. Therefore, the characteristic roots of P are all positive and less than one. Then (see Hirsch and Smale 1974), there is a norm $||\cdot||$ on $\mathbb{R}^4$, and $\mu \in (0, 1)$ such that:

$$||Pz|| \leq \mu ||z|| \text{ for all } z \in \mathbb{R}^4. \tag{72}$$

From (71), we get:

$$H(z_N) = P^s H(z_{N+s}) \text{ for } s = 1, 2, 3, \ldots \tag{73}$$

and using (72), we then obtain:

$$||H(z_N)|| \leq \mu^s ||H(z_{N+s})|| \text{ for } s = 1, 2, 3, \ldots \tag{74}$$

Clearly, $||H(z)||$ is a continuous function on the compact set $\mathbb{U}^4$, and has a maximum, call it $\mathbb{M}$. Because $H(z_N) \neq (0, 0, 0, 0)$, we have $||H(z_N)|| > 0$. We can choose S large enough so that $\mu^S\mathbb{M} < ||H(z_N)||$, Then, using (74), we get:

$$||H(z_N)|| \leq \mu^S||H(z_{N+S})|| \leq \mu^S\mathbb{M} < ||H(z_N)||,$$

a contradiction. Therefore, possibility (B) cannot occur either.

We can conclude that if (x_t) is an optimal program from some $(x, y) \gg 0$, such that $(x, y) \neq (\bar{x}, \bar{x})$, then the following condition cannot hold:

$$\lim_{t\to\infty} x_t = \bar{x}.$$

Now Theorem 2 follows directly from Theorem 1. □

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